

# Large Scale Structure Constraints for a Class of $f(R)$ Theories of Gravity

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Over the last few years much attention has been given to the study of modified gravity theories in order to find a more natural explanation for the late time acceleration of the Universe. Nevertheless, a comparison of the matter power spectrum predictions made by these theories with available data has not yet been subjected to a detailed analysis. In the context of  $f(R)$  theories of gravity we study the predicted power spectra using both a dynamical systems approach for the background and solving for the matter perturbations without using the quasi-static approximation, comparing the theoretical results with the SDSS data. The importance of studying the first order perturbed equations by assuming the correct background evolution and the relevance of the initial conditions are also stressed. We determine the statistical significance in relation to the observational data and demonstrate their conflict with existing observations.

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## I. INTRODUCTION

Despite more than a century of careful scrutiny, General Relativity (GR) remains the best fundamental theory of physics for describing gravitational interactions. When applied to cosmology, assuming that the large-scale geometry of the Universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric and adopting a fluid description for the matter content consisting of baryons, Cold Dark Matter (CDM) and radiation, GR gives rise to a set of field equations which can be solved exactly to give the simplest expanding Universe model - the so called Friedmann model, governing the dynamics of the cosmological scale-factor  $a(t)$ . This model has been remarkably successful, giving for example the correct abundances of the lightest elements and explaining the origin of the Cosmic Microwave Background Radiation (CMBR). In the last two decades, however, advances in observational cosmology appear to suggest that if one wishes to retain the FLRW metric, the Universe must have undergone two periods of accelerated expansion. The first period is needed to explain the flatness problem and the near-scale invariant spectrum of temperature fluctuations observed in the CMBR, while the second period explains the dimming of distant type Ia supernovae relative to Einstein-de Sitter Universe model. In order to explain these periods of acceleration, the strong energy condition ( $\mu + 3p \geq 0$ ) needs to be violated. In the case of inflation, this is achieved by introducing a dynamical scalar field, while the present day acceleration is most easily explained with the introduction of a positive cosmological constant. The resulting description of the Universe has become known as the  $\Lambda$ CDM or *Concordance Model* [1]. Although this beautifully simple phenomenological model appears to fit all currently available

observations (Supernovae Ia [2], CMBR anisotropies [3], Large Scale Structure formation [4], baryon acoustic oscillations [5] and weak lensing [6]), it is affected by significant fine-tuning problems related to the vacuum energy scale and this has led in recent years to efforts to explore alternatives to this description of the Universe.

The leading alternative to the  $\Lambda$ CDM model is based on modifications of the standard Einstein-Hilbert action. This is due to the fact that these changes naturally admit a phase of late time accelerated expansion (an early universe inflationary phase is also possible [7]). In this way Dark Energy can be thought of as having a geometrical origin, rather than being due to the vacuum energy or additional scalar fields which are added by hand to the energy momentum tensor (see [8] and references therein for an extensive presentation of the state of the art of this programme of investigation).

One of the simplest extensions of GR is based on gravitational actions which are non-linear in the Ricci curvature  $R$  and/or contain terms involving combinations of derivatives of  $R$  [8–14]. An important feature of these theories is that the field equations can be written in a way which makes it easy to compare with GR. This is done by moving all the higher-order corrections to the curvature onto the RHS of the field equations and defining an “effective” source term, known as the *curvature fluid*. Once this has been done the strong energy condition can be easily violated and this gives rise to a curvature fluid-driven period of late-time acceleration. Unfortunately this comes at the cost of having to study a considerably more complex set of field equations, making it difficult to obtain both exact and numerical solutions which can be compared with observations. Many studies of the expansion history of  $f(R)$  gravity have been performed using a range of strategies for numerically integrating the cos-

mological equations and these studies have highlighted, among other things, how sensitive results are to initial conditions, the presence of rip singularities and oscillations in the deceleration and snap parameters, the existence and stability of the Einstein Static models and bounce solutions [15].

These difficulties can be reduced somewhat by using the theory of dynamical systems [16], which, with careful choice of dynamical variables, provides a relatively simple method for obtaining exact solutions (via the equilibrium points of the system) and a description of the global dynamics of these models for a given  $f(R)$  theory [17].

Another useful (but more limited) approach is to assume that the expansion history of the Universe is known exactly, and to invert the field equations to deduce what class of  $f(R)$  theories give rise to this particular cosmological evolution [18]. This has been done recently for exact power-law solutions for the scale factor, corresponding to phases of cosmic evolution when the energy density is dominated by a perfect fluid. It was found that such expansion histories only exist for modifications of the type  $R^n$  [19]. It was also shown in [20] that the only  $f(R)$  theory of gravity that admits an exact  $\Lambda$ CDM expansion history is standard General Relativity with a positive cosmological constant and if one wants to obtain this behaviour of the scale factor for more general functions of  $R$ , additional degrees of freedom need to be added to the matter sector. A more extensive analysis of *reconstruction methods* has been carried out in [21] to obtain theories which give an approximate description of deceleration-acceleration transitions in cosmology and also in [22] where the reconstruction method was based on standard cosmic parameters instead of specifying the time evolution of the scale factor.

Because it is possible to find background expansion histories which are consistent with the standard  $\Lambda$ CDM model, it is necessary to investigate the growth of structure in order to break this degeneracy. This requires extending the standard theory of cosmological perturbations for GR to  $f(R)$  gravity. This has been done using both the metric based approach [23–26] originally developed for GR by Bardeen [27] and the 1 + 3 covariant approach first introduced by Ellis and Bruni [28]. For example, in [29], evolution equations were obtained for scalar and tensor perturbations for  $f(R)$  gravity and applied to the spatially flat, matter dominated solution of  $R^n$  gravity given by  $a(t) = a_0 t^{\frac{2n}{3(1+w)}}$  (which is a saddle point  $G$  of the corresponding dynamical system for these theories).

The results obtained demonstrate that the evolution of scalar perturbations is determined by a fourth order differential equation rather than a second order one, so that the evolution of density fluctuations contains, in general, four modes rather than two and can give rise to a more complex evolution than what is obtained in General Relativity. It was also found that the perturba-

tions depend on the scale for any value of the equation of state parameter of standard matter (while in GR the evolution of the dust perturbations is not scale dependent) and that there is a characteristic scale-dependent signature in the matter power spectrum [30]. It was also found that the growth of large density fluctuations can occur also in backgrounds in which the expansion rate is increasing in time. This surprising result is strikingly different from what one finds in GR and could provide a strong constraint on some  $f(R)$  theories using the integrated Sachs-Wolfe (ISW) effect and the matter power spectrum [31].

These features can be interpreted by comparing the system of fourth order equations, which produced them, with those governing scalar perturbations for two interacting fluids in GR [32], because they have the same structure, i.e., there are friction and source terms due to the interaction of the two effective fluids. On very large and on very small scales, the system of equations become independent of  $k$ , so that the evolution of the perturbations does not change as a function of scale and the power spectrum is consequently scale invariant. On intermediate scales the interaction between the two fluids is maximised and the curvature fluid acts as a relativistic component whose pressure is responsible for the oscillations and the dissipation of the small scale perturbations in the same way in which the photons operate in a baryon-photon system. This suggests that the variables describing the fluctuations in the curvature fluid can be interpreted as representing the modes associated with the additional scalar degree of freedom typical of  $f(R)$ -gravity. In this sense the spectrum can be explained physically as a consequence of the interaction between these scalar modes and standard matter and result in a considerable loss of power for a relatively small variation of the parameter  $n$ .

The results described above were obtained assuming the simple background model  $G$  and can therefore only provide hints about what features to expect in the matter power spectrum when realistic expansion histories are considered. In order to test the robustness of the results presented in [29], the complete expansion history of the background FLRW universe for  $R^n$  gravity, which resembles the  $\Lambda$ CDM model needs to be found. This is achieved by integrating the dynamical systems equations so that an orbit representing a cosmic history passes close to the matter dominated point  $G$  and eventually tends towards the late-time attractor  $C$ . In [17] it was shown that if  $1.36 < n < 1.5$ ,  $G$  and  $C$  respectively represent a decelerated matter dominated solution and late-time power-law acceleration. The equations for  $\Delta_m$  and  $\mathcal{R}$  can then be solved for this background to obtain a matter power spectrum which can be directly compared with the one found for  $\Lambda$ CDM.

This paper is organised as follows: In Section II we give the basic equations governing the background evo-

lution of FLRW models and briefly present the key variables needed to study cosmological perturbations in  $f(R)$  gravity using the covariant approach. In Section III we discuss the cosmology of  $R^n$  gravity and describe their main features by recasting the cosmological equations as an autonomous system of first order equations. We then integrate these equations for different values of the parameter  $n$  using initial conditions in the matter dominated epoch which have Hubble and deceleration parameters equal to their  $\Lambda$ CDM values. For such initial conditions, we find that it is impossible to have cosmic histories that simultaneously have present day values of the Hubble and deceleration parameters close to their  $\Lambda$ CDM values today. In Section IV we determine the matter power spectra for the expansion histories given in V and compare them with what is obtained by integrating the perturbation equations for the matter dominated solution  $G$ . We find that although the broad large and small scale features of the power spectrum are largely the same as in [30], the scale-dependent features are no longer present when the complete background expansion history is considered. We then use the observed matter power spectrum based on the luminous red galaxies in the Sloan Digital Sky Survey (SDSS) [33] to directly constrain this class of  $f(R)$  theories of gravity and find that only for very special initial conditions, the models considered give power spectra in the Sloan wavenumber interval, which are in good agreement with the available data for luminous red galaxies.

Finally in Section V we discuss the results and give an outline of future work to be done.

Unless otherwise specified, natural units ( $\hbar = c = k_B = 8\pi G = 1$ ) will be used throughout this paper, Latin indices run from 0 to 3. The symbol  $\nabla$  represents the usual covariant derivative and we use the  $-$ ,  $+$ ,  $+$ ,  $+$  signature and the Riemann tensor is defined by

$$R^a{}_{bcd} = W^a{}_{bd,c} - W^a{}_{bc,d} + W^e{}_{bd}W^a{}_{ce} - W^f{}_{bc}W^a{}_{df}, \quad (1)$$

where the  $W^a{}_{bd}$  are the Christoffel symbols (i.e., symmetric in the lower indices), defined by

$$W^a{}_{bd} = \frac{1}{2}g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \quad (2)$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd}R_{acbd}. \quad (3)$$

Finally the action for  $f(R)$ -gravity can be written in these units as:

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ \frac{1}{2}f(R) + \mathcal{L}_m \right], \quad (4)$$

where  $R$  is the Ricci scalar,  $f$  is general differentiable (at least  $C^2$ ) function of the Ricci scalar and  $\mathcal{L}_m$  corresponds to the matter Lagrangian.

## II. THE COSMOLOGICAL EQUATIONS FOR $f(R)$ GRAVITY

### The Background FLRW equations

In a FLRW universe, the non-trivial field equations lead to the following equations governing the expansion history of the Universe:

$$3\dot{H} + 3H^2 = -\frac{1}{2f'} \left[ \mu_m + 3p_m + f - f'R + 3Hf''\dot{R} + 3f'''\dot{R}^2 + 3f''\ddot{R} \right], \quad (5)$$

$$3H^2 = \frac{1}{f'} \left[ \mu_m + \frac{Rf' - f}{2} - 3Hf''\dot{R} \right], \quad (6)$$

i.e., the *Raychaudhuri* and *Friedmann* equations [34]. Here  $H$  is the Hubble parameter, which defines the scale factor  $a(t)$  via the standard relation  $H = \dot{a}/a$ , the Ricci scalar is

$$R = 6\dot{H} + 12H^2 \quad (7)$$

and  $f'$ ,  $f''$  and  $f'''$  abbreviate  $\partial^n f / (\partial R)^n$  for  $n = 1..3$  respectively. The *energy conservation equation* for standard matter

$$\dot{\mu}_m = -3H\mu_m(1+w) \quad (8)$$

closes the system, where  $w$  is the barotropic equation of state.

Note that the Raychaudhuri equation can be obtained from the Friedmann equation, the energy conservation equation and the definition of the Ricci scalar. Hence, any solution of the Friedmann equation automatically solves the Raychaudhuri equation.

### Density perturbations in $f(R)$ gravity

Density (scalar) perturbations may be extracted from any first order tensor  $T_{ab}$  orthogonal to the 4-velocity  $u^a$  by using a *local* decomposition [28], so that repeated application of the operator  $D_a \equiv h^b{}_a \nabla_b$  on  $T_{ab}$  extracts the scalar part of the perturbation variables. In this way we can define the following scalar quantities

$$\begin{aligned} \Delta_m &= \frac{a^2}{\mu_m} D^2 \mu_m, & Z &= 3a^2 D^2 H, & C &= S^4 D^2 \tilde{R}, \\ \mathcal{R} &= a^2 D^2 R, & \mathfrak{R} &= a^2 D^2 \dot{R}, \end{aligned} \quad (9)$$

where  $h_{ab}$  is the projection tensor into the rest-spaces orthogonal to  $u^a$ .  $\Delta_m^m$ ,  $Z$  respectively represent the fluctuations in the matter energy density  $\mu_m$  and expansion  $\Theta$ , and  $\mathcal{R}$ ,  $\mathfrak{R}$  determine the fluctuations in the Ricci scalar  $R$  and its momentum  $\dot{R}$ . This set of variables completely characterises the evolution of density perturbations. Then, using eigenfunctions of the spatial Laplace-Beltrami operator defined in [28]:  $D^2 Q = -\frac{k^2}{a^2} Q$ , where

$k = 2\pi a/\lambda$  is the wavenumber and  $\dot{Q} = 0$ , we can expand every first order quantity in the above equations, so for example in the case of  $\Delta_m$  we have

$$\Delta_m(t, \mathbf{x}) = \sum \Delta_m^{(k)}(t) Q^{(k)}(\mathbf{x}), \quad (10)$$

where  $\sum$  stands for both a summation over a discrete index or an integration over a continuous one. In this

way, it is straightforward, although lengthy, to derive a pair of second order equations describing the  $k^{th}$  mode for density perturbations in  $f(R)$  gravity. They are:

$$\begin{aligned} \ddot{\Delta}_m^k - \left[ (3w-2)H + \frac{\dot{R}f''}{f'} \right] \dot{\Delta}_m^k + \left[ w\frac{k^2}{a^2} + (w-1)\frac{\mu_m}{f'} - w\frac{f}{f'} \right] \Delta_m^k \\ = \frac{1+w}{2} \left[ -1 - \frac{2k^2}{a^2} \frac{f''}{f'} + (f - 2\mu_m + 6\dot{R}Hf'') \frac{f''}{f'^2} - 6\dot{R}H \frac{f'''}{f'} \right] \mathcal{R} - \frac{3(1+w)}{f'} H f'' \dot{\mathcal{R}}^k, \\ \ddot{\mathcal{R}}^k + \left( 2\dot{R} \frac{f'''}{f''} + 3H \right) \dot{\mathcal{R}}^k + \left[ \frac{k^2}{a^2} + \ddot{R} \frac{f'''}{f''} + \dot{R}^2 \frac{f^{(iv)}}{f''} + 3H\dot{R} \frac{f'''}{f''} + \frac{f'}{3f''} - \frac{R}{3} \right] \mathcal{R}^k \\ = - \left[ \frac{1}{3}(3w-1)\frac{\mu_m}{f''} + \frac{w}{1+w} \left( 2\ddot{R} + 2\dot{R}^2 \frac{f'''}{f''} + 6\dot{R}H \right) \right] \Delta_m^k + \frac{1-w}{1+w} \dot{R} \dot{\Delta}_m^k. \end{aligned} \quad (11)$$

Already on super-Hubble scales,  $k/aH \ll 1$ , a number of important features can be found which allow one to differentiate from what is obtained in GR [29]. Firstly, it is clear that the evolution of density perturbations is determined by a *fourth order* differential equation rather than a second order one. This implies that the evolution of the density fluctuations contains, in general, four modes rather than two and can give rise to a more complex evolution than the one of GR. Secondly, the perturbations are found to depend on the scale for any equation of state for standard matter (while in GR the evolution of the CDM perturbations are scale-invariant). This means that even for dust, the evolution of super-horizon and sub-horizon perturbations are different. Thirdly, it is found that the growth of large density fluctuations can occur also in backgrounds in which the expansion rate is increasing in time. This is in striking contrast with what one finds in GR and would lead to a time-varying gravitational potential, putting tight constraints on the ISW for these models.

Let us now turn to the case of a general wave mode  $k$ . One of the most instructive ways of understanding the details of the evolution of density perturbations for a general  $k$  is to compute the matter transfer function  $T(k)$ , defined by the relation [35]  $\langle \Delta_m(\mathbf{k}_1) \Delta_m(\mathbf{k}_2) \rangle = T(k_1) \Delta(\mathbf{k}_1 + \mathbf{k}_2)$ , where  $\mathbf{k}_i$  are two wavevectors characterizing two Fourier components of the solutions of (11) and  $T(\mathbf{k}_1) = T(k_1)$  because of isotropy in the distribution of the perturbations. This quantity tells us how the fluctuations of matter depend on the wavenumber at a specific time and carries information about the amplitude of the perturbations (but not on their spatial structure).

In GR, the transfer function on large scales is constant, while on small scales it is suppressed in comparison with the large scales (i.e., modes which entered the horizon during the radiation era) [36]. In the case of pure dust in GR the transfer function is scale invariant. Substituting the details of the background, the values of the parameter  $n$ , the barotropic factor  $w$  and the wavenumber  $k$  into (11) one is able to obtain  $T(k)$  numerically.

One can easily see from expressions (11) that the matter power-spectrum in  $f(R)$  gravity theories is further processed after equality and would differ from the standard  $\Lambda$ CDM power spectrum  $P_k^{\Lambda\text{CDM}}$  when evaluated today. The latter is widely assumed to represent accurately the evolution of perturbations till radiation-matter equality, since before that the effects of any modification to the usual Concordance Model needs to be negligible in order to preserve the cosmological standard model predictions in the radiation-dominated epoch such as the primordial light elements abundances during Big Bang Nucleosynthesis.

Therefore, these two power spectra, when evaluated today, would be related linearly by a transfer function  $T(k)$  given by

$$P_k^{f(R)} = T(k) P_k^{\Lambda\text{CDM}}|_{eq} \quad (12)$$

where  $T(k) \propto |\Delta_m^k|_{today}^2$  and  $\Delta_m^k$  is obtained from the system of equations (11).

On linear scales,  $P_k^{f(R)}$  will in general depend on both the  $f(R)$ -model and the scale  $k$ , therefore differing from the  $\Lambda$ CDM model, where it is scale-invariant.

In the GR limit:  $f(R) = R$ , (11) reduces to the standard equations for the evolution density perturbations in



GR:

$$\ddot{\Delta}_m^k - (3w - 2)H\dot{\Delta}_m^k + \left[ w\frac{k^2}{a^2} + \left( \frac{-1 + 2w - 3w^2}{2} \right) \mu_m \right] \Delta_m^k = 0, \quad (13)$$

$$\mathcal{R}^k = (1 - 3w)\mu_m\Delta_m^k, \quad (14)$$

and one can easily see that the linear evolution of CDM density perturbations for sub-Hubble ( $k \gg aH$ ) scales in  $\Lambda$ CDM is given by the well-known result:

$$\Delta_m''^k + \mathcal{H}\Delta_m'^k - \frac{1}{2}a^2\mu_m\Delta_m^k = 0, \quad (15)$$

where  $\mathcal{H} = a'/a$  and prime (only for this equation) denotes derivative with respect to conformal time.

Notice that according to (15) the evolution of the Fourier modes does not depend upon  $k$ . This means that for  $\Lambda$ CDM on sub-Hubble scales, once the density contrast starts to grow after matter-radiation equality, evolution only changes the overall normalisation of the matter power-spectrum  $P(k)$ , but not its shape.

### III. DETERMINING THE EXPANSION HISTORY FOR $R^n$ -GRAVITY

To proceed, we need to fix our theory of gravity. The simplest and most widely studied form of  $f(R)$  gravitational theories is  $f(R) = \alpha H_0^2 (R/H_0^2)^n$ , where  $\alpha = \alpha(n)$  is a non-dimensional coupling constant and  $H_0$  is the  $\Lambda$ CDM value of the Hubble parameter today. For this class of models the cosmological equations associated with a FLRW universe are particularly easy to analyse.

However the aim of this investigation is to show that studies of different  $f(R)$ -gravity models that share similar background expansion history with  $\Lambda$ CDM can in principle provide useful constraints on the viability of these models via the power spectra of matter density perturbations they produce.

The first step in the implementation of the Dynamical Systems (DS) approach for determining the expansion history for  $R^n$  gravity is the definition of the key DS variables. Following [17], we introduce the dimensionless variables:

$$x = \frac{\dot{R}(n-1)}{HR}, \quad y = \frac{R(1-n)}{6nH^2}, \quad \Omega_d = \frac{\mu_d}{3n\alpha H^2 R^{n-1}} \quad (16)$$

In terms of these variables, the Friedmann equation (6) takes the form

$$1 + x + y - \Omega_d = 0. \quad (17)$$

An autonomous system of ordinary differential equations, which are equivalent to cosmological equations (5-8) can

be obtained by differentiating (16) with respect to redshift  $z$ . Here we give the equations for dust ( $w = 0$ ), while those for a general barotropic equation of state  $w$  are presented in [17]:

$$\begin{aligned} -(z+1)\frac{dx}{dz} &= -x - x^2 + \frac{(4-2n+nx)y}{n-1} + \Omega_d, \\ -(z+1)\frac{dy}{dz} &= 4y + \frac{(x+2ny)y}{n-1}, \\ -(z+1)\frac{d\Omega_d}{dz} &= \left(1 - x + \frac{2ny}{n-1}\right)\Omega_d. \end{aligned} \quad (18)$$

The dimensionality of the resultant system (18) can be reduced further using the Friedmann constraint (17). The evolution of the Hubble parameter can then be determined by writing (3) in terms of the DS variables:

$$(1+z)\frac{dh}{dz} = \frac{h(2+ny)}{n-1}, \quad (19)$$

where  $h = H/H_0$ . Furthermore the deceleration parameter can be determined directly from  $y$ :

$$q = \frac{ny}{(n-1)} + 1. \quad (20)$$

In [17] it was shown that these equations admit a number of fixed points of which two are particularly interesting. The points, labeled  $G$  and  $C$  in [17] correspond to two cosmologically interesting exact solutions:  $G$  corresponds to a matter dominated saddle point which in the case of dust has  $a = a_0 t^{2n/3}$  and  $C$  is the late-time attractor with  $a = a_0 t^{\frac{(1-n)(2n-1)}{n-2}}$ . In [17] it was also shown that  $C$  and  $G$  respectively represent decelerated and accelerated phases of the Universe with positive energy density if  $n$  lies in the range  $1.36 < n < 1.5$ .

With this in mind let us integrate (18), by fixing the initial conditions for the DS variables (16) to be identical to their  $\Lambda$ CDM values deep in the dust dominated era (at a redshift  $z_0 = 2000$ ) and determine the expansion history for  $R^n$  models with eight different values for the exponent  $n > 1$  between  $n = 1.1$  and  $1.4$  in order to allow the possibility of late-time acceleration. In this way we can determine for which values of  $n$  we obtain present day values for  $q(z)$  and  $H(z)$  consistent with the  $\Lambda$ CDM model. It is clear from the results in Table I that

$n$	1.1	1.2	1.27	1.29	1.3	1.31	1.33	1.4
$h_0$	0.63	0.73	0.94	0.95	1.08	1.44	3.27	45.03
$q_0$	0.38	0.21	0.15	0.13	0.22	0.27	0.21	-0.17

Table I: Present-day values of the Hubble  $h_0 \equiv H(\text{today})/H_0$  and deceleration ( $q_0$ ) parameters for the  $R^n$  models under consideration.  $H_0$  corresponds to the  $\Lambda$ CDM Hubble parameter value today. Only  $n = 1.4$  provides acceleration at the present time, whereas  $n = 1.29$  gives the closest value for  $h_0$  to  $\Lambda$ CDM.

it is not possible for  $R^n$  gravity to admit FLRW cosmic

$n$ exponent	1.1	1.2	1.27	1.29	1.3	1.31	1.33	1.4
$\chi^2$	15.1394	13.1839	13.0184	13.0093	13.0104	13.0098	13.0102	13.0128
$\sigma$ exclusion	1.9849	1.4086	1.3486	1.3452	1.3457	1.3454	1.3456	1.3465
% suppression	29.5	7.94	4.75	4.45	4.37	4.33	4.35	4.47

Table II: Fits to the SDSS data for  $R^n$  cosmology by using set of initial conditions **III**: eight different values of exponent  $n$  were investigated from  $n = 1.1$  to  $1.4$ . Values for  $\chi^2$  and the confidence region  $\sigma$  are presented in the second and third rows respectively. The data to be fitted by the theoretical spectra are taken from [33] and normalisation from WMAP7 was imposed for all the studied models. The fit provided by  $\Lambda$ CDM ( $\chi^2 = 11.1996$ ) is not improved by any of these parameter values. The final row gives the suppression in the overall initial amplitude required to get the best fits. For all the values, this suppression turns out to be smaller than 30% and is therefore in the experimental uncertainty interval for this quantity. One can see that the best fit corresponds to the value  $n = 1.29$  with a suppression of 4.45% and good fits are also obtained for  $n = 1.3$ ,  $1.31$  and  $1.33$  with similar suppressions.

histories that simultaneously have present-day values of the Hubble and deceleration parameters which are close to their  $\Lambda$ CDM values today if initial conditions are chosen in order that the expansion history is close to the  $\Lambda$ CDM model at early times. Even without considering LSS data, this already put severe constraints on these models.

#### IV. THE MATTER POWER SPECTRUM AND SDSS CONSTRAINTS

Let us now turn to the matter power spectrum.

Taking the dominant component to be dust, the system (11) can be written in terms of the dynamical variables:

$$\begin{aligned}
& \frac{(-1+n)^2(1+z)}{2ny} \hat{\mathcal{R}}^{k'} - \frac{3h^2 [(-1+n)(1+(-2+n)\Omega_d) + (-2+n)y] + \hat{k}^2(-1+n)^2(1+z)^2}{6h^2ny} \hat{\mathcal{R}}^k \\
& + h^2(1+z)^2 \Delta_m^{k''} + h^2 \frac{[(-1+n)\Omega_d + y](1+z)}{n-1} \Delta_m^{k'} - 3h^2 \Omega_d \Delta_m^k = 0, \\
& \hat{\mathcal{R}}^{k''} - \frac{[4-4\Omega_d+4y+n(-2+2\Omega_d-3y)]}{(-1+n)(1+z)} \hat{\mathcal{R}}^{k'} + \left\{ \frac{\hat{k}^2}{h^2} + \frac{(-2+n)[- \Omega_d^2 - (-1+y)^2 + \Omega_d(1+n+2y)]}{(-1+n)^2(1+z)^2} \right\} \hat{\mathcal{R}}^k \\
& + \frac{6h^2ny(1-\Omega_d+y)}{(-1+n)^2(1+z)} \Delta_m^{k'} + \frac{6h^2n\Omega_dy}{(-1+n)^2(1+z)^2} \Delta_m^k = 0, \tag{21}
\end{aligned}$$

where prime denotes derivative with respect to redshift and the dimensionless quantities  $\hat{\mathcal{R}}^k = \mathcal{R}^k/H_0^2$  and  $\hat{k} = k/H_0$  have been introduced. Note that equations (21) are valid only for  $n \neq 1$ .

SDSS correlation data from LRG have been used to test the predictions from the  $\Lambda$ CDM power spectrum obtained from linear perturbation theory with WMAP cosmological data to high accuracy ( $\chi^2 \approx 11.2$ , degrees of freedom (*d.o.f.*) = 14) [33]. In what follows, we will do the same for this class of  $f(R)$  theories of gravity.

To do this we first determine the cosmological background evolution as described in the previous section and then use these results to solve the system of equations (21) in order to obtain the density contrast today. Then, by applying expression (12) to these results, one

can obtain the fully processed power spectra  $P_k^{f(R)}$  for the above models, which can be compared to the  $\Lambda$ CDM predictions and the LRG data.

Before proceeding, let us mention that three sets of different initial conditions were considered for the system (21) in order to determine how sensitive the final processed power spectrum is to changes in these values:

- **I**:  $\Delta_m^k|_0 = \hat{\mathcal{R}}^k|_0 = 10^{-5}$ ,  $\Delta_m^{k'}|_0 = \hat{\mathcal{R}}^{k'}|_0 = 10^{-5}$ ,
- **II**:  $\Delta_m^k|_0 = \hat{\mathcal{R}}^k|_0 = 10^{-5}$ ,  $\Delta_m^{k'}|_0 = \hat{\mathcal{R}}^{k'}|_0 = 10^{-8}$ ,
- **III**:  $\Delta_m^k|_0 = \hat{\mathcal{R}}^k|_0 = 10^{-5}$ ,  $\Delta_m^{k'}|_0 = \hat{\mathcal{R}}^{k'}|_0 = 0$ ,

where the subscript 0 refers to the initial redshift  $z_0 =$

2000. The choice of sets **I** and **II** as initial conditions for

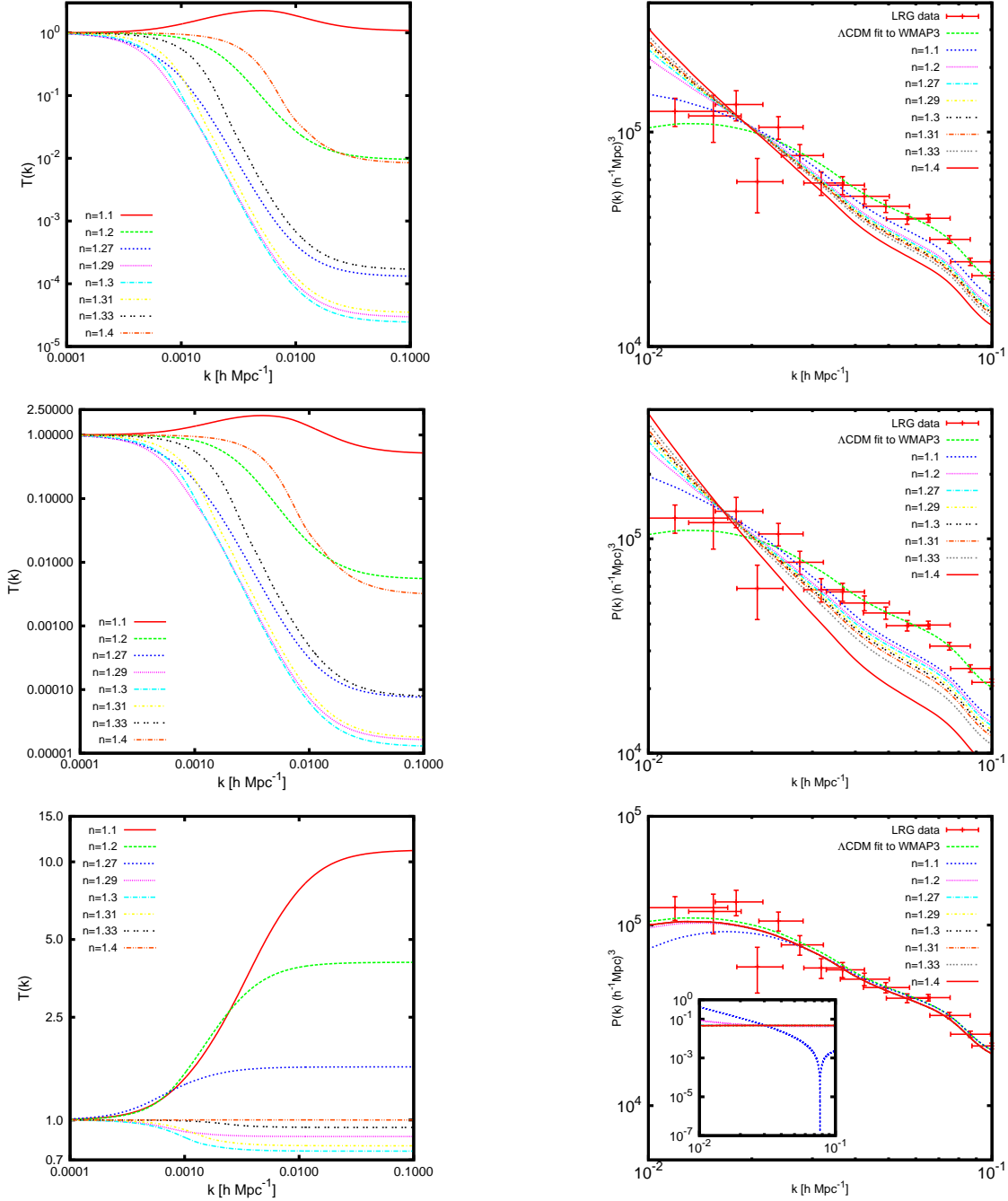


Figure 1: The left panels show the transfer function  $T(k) = |\Delta_k / \Delta_k^{\Lambda\text{CDM}}(z = 2000)|^2$  evaluated today ( $z = 0$ ) for wavenumber  $k$  (in  $h \text{ Mpc}^{-1}$  units) in the range 0.0001 to 0.1 for the initial conditions sets **I**, **II** and **III** as described in the bulk of this investigation. The transfer functions  $T(k)$  on left panels have been normalised in such a way that the curves coincide on large scales. One can see how for conditions **I** and **II** the power spectra are hardly distinguishable whereas for the conditions **III** the results differ. On the right panels we present the corresponding linear matter power-spectra  $P(k)$  for  $\Lambda$ CDM and  $R^n$  models for  $n = 1.1, 1.2, 1.27, 1.29, 1.3, 1.31, 1.33$  and 1.4 data correspond to SDSS data [33]. All the power spectra were assumed to have an arbitrary overall normalisation at the scale  $k = 0.01 h \text{ Mpc}^{-1}$  in order to find the best fit to the data. Conditions **I** and **II** lead to power spectra in complete disagreement with the observed data. Conditions **III**, due to the almost flatness of the spectra in the range covered by Sloan data present a good fit to the data. On the right bottom panel we show in a window the relative discrepancy between the  $\Lambda$ CDM and the  $R^n$  fits power-spectra for every studied exponent. The smallest discrepancy in scales  $k > 3 \cdot 10^{-2}$  happens for  $n = 1.1$  whereas for smaller scales, all the remaining values of  $n$  provide similar relative error around  $5 \cdot 10^{-2}$ .

the system (11) can be understood as providing scale-invariant initial conditions for the variables  $\Delta_m^k$  and  $\hat{\mathcal{R}}^k$  and their first derivatives are all taken to be small (but non-zero) at the initial redshift. On the other hand, set **III** we set the first derivatives of  $\Delta_m^k$  and  $\hat{\mathcal{R}}^k$  to zero at  $z = z_0$ . This choice has important consequences for the obtained spectra.

For each value of  $n$ , we present in Fig. 1 both the transfer functions and the processed power spectra for all the initial conditions sets just mentioned.

One can see how for these models that the transfer functions have a nearly-flat plateau on large scales [30] regardless of the set of initial conditions. On intermediate scales however, the density contrast behaviour and its amplitudes today depend both upon the value of  $n$  and the initial conditions. The left panels in Fig. 1 clearly illustrate this.

Then, by using expression (12) for the transfer function, one can then obtain the processed power spectra and compare these theoretical results with the LRG data. We find that initial conditions **I** and **II** are not able to provide good fits to the SDSS data due to the fact that in the required scales ( $0.01 - 0.1 \text{ h Mpc}^{-1}$ ) the spectra are not flat but decrease with the wavenumber  $k$  (see the left panels in Fig. 1). On the other hand, initial conditions **III** give power spectra which are in good agreement with the LRG data. This is due to the almost-flat transfer function in the LRG data range. Note however that the initial amplitude was assumed to be a free parameter which was determined to achieve the best fit.

For this set of initial conditions, Table II give the  $\chi^2$  (*d.o.f.* = 14) analysis for the eight studied  $R^n$  models when their respective spectra evolutions are fitted to the SDSS data, assumed to be non-correlated. We also include the value for the confidence regions  $\sigma$  with respect to  $\Lambda$ CDM as well as the overall amplitude suppression in the initial scale ( $k = 0.01 \text{ h Mpc}^{-1}$ ) to get the best fits after a Least Square method analysis. It is clear that none of the  $R^n$  models under consideration acquire the same goodness of fit as the  $\Lambda$ CDM model as seen in Table II in the  $\sigma$  exclusion regions.

For completeness and in order to emphasise the importance of using the complete expansion history for the background, we have also given  $|\Delta_m^k|^2$  for models whose background evolution is given by the exact saddle point solution  $G$  in the case of dust ( $w = 0$ ). We do this for the same parameter values  $n$  and initial conditions **I**, **II** and **III** as shown in Fig. 2. These results agree with previous investigations [30] that showed how when this background scale factor is assumed the spectrum is composed of three parts corresponding to three different evolution regimes for the perturbations. In this scenario, on intermediate scales the interaction between the two fluids (dust and curvature) is maximised and the curvature fluid acts as a relativistic component whose pressure is responsible for the oscillations and the dissipation of

the small scale perturbations in the same way in which the photons operate in a baryon-photon system [30]. If we compare these results to the left panel in Fig. 1, we conclude that the scale-dependent features in Fig. 2 are washed out when the complete background expansion history is considered, however the main large and small scale features of the power spectrum found in [30] are retained.

## V. DISCUSSION AND FUTURE WORK

In this paper we presented a complete analysis of the background and matter perturbations for one of the most widely studied modified gravity theories:  $R^n$  gravity with  $n \gtrsim 1$ . Both the cosmological background evolution and linear perturbation equations were solved by combining the dynamical systems approach for the background and using the 1 + 3 covariant approach to evolve the matter perturbations, without assuming any intermediate (quasi-static) approximation.

We solved the background equations for different values of the parameter  $n$  using initial conditions in the matter dominated epoch, with Hubble and deceleration parameters equal to their  $\Lambda$ CDM values. For such initial conditions, we found that it is impossible to have cosmic histories that simultaneously have present day values of these cosmological parameters close to their  $\Lambda$ CDM values today. In fact, of the eight models considered, only  $n = 1.4$  provided a negative deceleration parameter today, but gave a Hubble parameter completely incompatible with its observed value, while  $n = 1.29$  gave the closest value for the present-day Hubble parameter to  $\Lambda$ CDM but exhibits no late time acceleration.

We then used the observed matter power spectrum based on the luminous red galaxies in the Sloan Digital Sky Survey to further constrain these models. For the studied exponents, we found that all the models gave rise to an almost-flat transfer functions in the Sloan wavenumber interval provided very special initial conditions are chosen. In this case the best fit to the data was found for the value  $n = 1.29$  with a suppression of 4.45% and good fits were also obtained for  $n = 1.3$ , 1.31 and 1.33. The exponent  $n = 1.4$  (the only one providing acceleration today) required a suppression slightly bigger (4.47%). None of the studied exponents were however able to fit the data as well as the  $\Lambda$ CDM model.

It is clear from this analysis that  $R^n$  gravity does not successfully meet any of the cosmology requirements for it to be considered as a viable alternative to the standard model. This work does however illustrate the utility of our approach and it should be possible to use these techniques to constrain which  $f(R)$  theories remain consistent with current data even if they are indistinguishable from the  $\Lambda$ CDM model at the level of the FLRW background.

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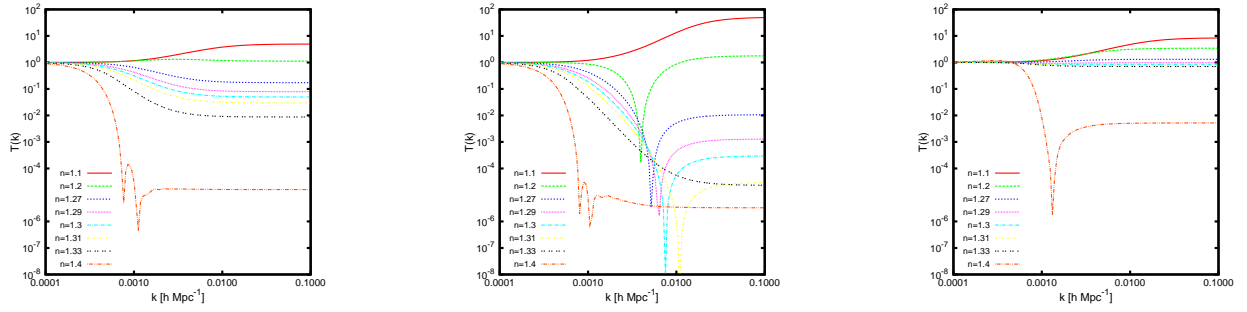


Figure 2: Saddle-point analysis: scale dependence of the transfer function  $T(k) = |\Delta_k/\Delta_k^{\Lambda\text{CDM}}(z=2000)|^2$  evaluated today ( $z=0$ ) for wavenumber  $k$  (in  $h\text{Mpc}^{-1}$  units) in the range 0.0001 to 0.1 for the initial same conditions **I**, **II** and **III** used in the rest of this investigation. The transfer functions have been normalized in such a way that the curves coincide on large scales. Both on very large and on very small scales, the  $\Delta$  becomes  $k$ -independent, so that the evolution of the perturbations does not change as a function of scale and the transfer function is consequently scale invariant. On intermediate scales the curvature fluid causes the oscillations. The result is a considerable loss of power for a relatively small variation of the parameter  $n$ . As before the shape and amplitudes (i.e. the increasing or decreasing behaviour with  $k$ ) of the transfer functions depend on the initial conditions **I** (left), **II** (center) and **III** (right). For example, when  $n=1.27$  we have decreasing behaviour for **I** and **II** but increasing behaviour for **III**.

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